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ON THE THEORY OF THE INSTABILITY OF THE TWO-DIMENSIONAL STAGNATION-POINT FLOW

By Günther Hämmerlin

Translation

"Zur Instabilitätstheorie der ebenen Staupunktströmung."
50 Jahre Grenzschichtforschung. Friedr. Vieweg &
Sohn (Braunschweig), 1955, pp. 315-327.

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ON THE THEORY OF THE INSTABILITY OF THE TWO-DIMENSIONAL STAGNATION-POINT FLOW*

By Günther Hämmerlin

Abstract. - In this work we are concerned with the solution of the differential equations (1) to which H. Görtler was led by his assumption of a three-dimensional instability of the two-dimensional stagnation-point flow (see preceding paper). These differential equations can be solved in a region $\eta \geq \eta_o$ where, for the function F occurring in the coefficients, its asymptotic form may be substituted with the boundary conditions satisfied at infinity. In order to determine the eigenvalues the solutions for $\eta \geq \eta_o$ and $\eta \leq \eta_o$ must agree at the point $\eta = \eta_o$. We first consider the neutral disturbances ($\beta = 0$). By the analytical method it was found that all $\bar{\alpha}^2$ values in the interval $0 < \bar{\alpha}^2 < 1$ are eigenvalues. With the program controlled Z 4 computer at the E.T.H. Zürich, the interval was further scanned for eigenvalues up to $\bar{\alpha}^2 = 5$. However, no additional eigenvalues were found. Further, it was shown that in addition to the neutral disturbances, true unstable disturbances also exist.

I. STATEMENT OF THE PROBLEM

In the eigenvalue problem, which leads to the assumption of a three-dimensional instability of the stagnation-point flow, we are dealing with the system of differential equations (as shown by H. Görtler in the preceding paper).[1]

$$L[u] = 3F'u - F''v$$
$$L[v'' - \bar{\alpha}^2 v] = L[u'] \tag{1}$$

*"Zur Instabilitätstheorie der ebenen Staupunktströmung." 50 Jahre Grenzschichtforschung, Friedr. Vieweg & Sohn (Braunschweig), 1955, pp. 315-327.

[1]For simplicity, the disturbance amplitude functions denoted as u_1 and v_1 by Görtler are here denoted as u and v.

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with the operator

$$L = \frac{d}{d\eta}\left\{\frac{d}{d\eta} + F\right\} - \bar{\alpha}^2 \tag{2}$$ [2]

These differential equations are to be solved under the following boundary conditions:

$$u(0) = v(0) = v'(0) = 0,$$

$$u(\infty) = 0, \; v(\eta) = 0(\eta) \quad \text{for} \quad \eta \to \infty, \; v'(\infty) = 0$$

We are interested in the solutions for real positive values of $\bar{\alpha}$ (i.e., positive values of the eigenvalue parameter $\bar{\alpha}^2$).[3]

$F(\eta)$ is a numerically given function, for which in $\eta \geq \eta_o$ $(\eta_o = 4)$:

$$\begin{aligned} F(\eta) &= \eta - c \qquad c = 0.6479004 \\ F' &= 1 \\ F'' &= 0 \end{aligned} \tag{2}$$

[NACA note: There are two equations numbered (2) in the original document.] Furthermore, $F(0) = F'(0) = 0$ and $F(\eta)$ monotonically approaches the asymptotic form (eq. (2)).

When written, the system (1), after simple transformation, reads

$$u'' + Fu' - (2F' + \bar{\alpha}^2)u = -F''v$$

$$\begin{aligned} u^{IV} + Fv''' + (F' - 2\bar{\alpha}^2)v'' - (\bar{\alpha}^2F - F'')v' + \\ \left[\bar{\alpha}^2(\bar{\alpha}^2 - F') + F''\right]v = 2(F'u)' \end{aligned} \tag{3}$$

II. THE SOLUTION $u(\eta)$

We shall first satisfy the equations (1) in the interval $\eta \geq \eta_o$ for the boundary conditions satisfied at infinity. The requirement that these solutions (outer solutions), agree with the solutions in the

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[2]First, we consider the neutral disturbances $(\beta = 0)$, and return to the unstable disturbances $(\beta > 0)$ in section VIII.

[3]Namely, we have $\bar{\alpha} = \frac{2\pi}{\lambda}\sqrt{\frac{\nu}{a}}$ (see H. Görtler).

interval $\eta \leq \eta_0$ (inner solutions) at the point $\eta = \eta_0$, will give us a condition for the eigenvalues $\bar{\alpha}^2$.

Setting

$$g = v'' - \bar{\alpha}^2 v - u'$$

and

$$\eta - c = \gamma, \ ' = \frac{d}{d\gamma}$$

we write the following equations (1) for $\gamma \geq \gamma_0$ (corresponding to $\eta \geq \eta_0$):

$$u'' + \gamma u' - (\bar{\alpha}^2 + 2)u = 0$$

$$g'' + \gamma g' - (\bar{\alpha}^2 - 1)g = 0$$

$$v'' - \bar{\alpha}^2 v = g + u' \tag{4}$$

$$u(\infty) = 0 \qquad v(\gamma) = o(\gamma) \quad ^4$$

$$v'(\infty) = 0$$

The first two of equations (4) are of the type

$$y'' + axy' + by = 0$$

According to reference 1, p. 475 (10) the general solution of this differential equation is

$$y = x^{-\frac{1}{2}} e^{-\frac{1}{4}ax^2} \bar{y}\left(\frac{b}{2a} - \frac{1}{4}; \frac{1}{4}; \frac{a}{2} x^2\right)$$

where

$$\bar{y}(k,m,z) = C_1 M_{k,m}(z) + C_2 M_{k,-m}(z)$$

[4]In this report the symbol o( ) refers to the behavior at infinity.

and where, as usual,

$$M_{k,m}(z) = z^{\frac{1}{2}+m} e^{-\frac{1}{2}z} {}_1F_1\left(\frac{1}{2} + m - k;\ 2m + 1;\ z\right)$$

$$M_{k,-m}(z) = z^{\frac{1}{2}-m} e^{-\frac{1}{2}z} {}_1F_1\left(\frac{1}{2} - m - k;\ -2m + 1;\ z\right)$$

with

$${}_1F_1(\rho;\sigma;z) = 1 + \frac{\rho}{\sigma} z + \frac{\rho(\rho + 1)}{\sigma(\sigma + 1)} \frac{z^2}{2!} + \frac{\rho(\rho + 1)}{\sigma(\sigma + 1)} \frac{(\rho + 2)}{(\sigma + 2)} \frac{z^3}{3!} +$$

denoting the Kummer confluent hypergeometric function.

Setting

$$a = 1,\ b = -(\bar{\alpha}^2 + 2),$$

$$k = -\frac{\bar{\alpha}^2 + 2}{2} - \frac{1}{4},\ m = \frac{1}{4},\ z = \frac{1}{2}\gamma^2$$

the general solution of the first of the differential equations (4) is thus found to be

$$u(\gamma) = \gamma^{-\frac{1}{2}} e^{-\frac{1}{4}\gamma^2} \left\{ C_1\left(\frac{1}{2}\gamma^2\right)^{3/4} e^{-\frac{1}{4}\gamma^2} {}_1F_1\left(\frac{\bar{\alpha}^2 + 4}{2};\ \frac{3}{2};\ \frac{1}{2}\gamma^2\right) + \right.$$

$$\left. C_2\left(\frac{1}{2}\gamma^2\right)^{1/4} e^{-\frac{1}{4}\gamma^2} {}_1F_1\left(\frac{\bar{\alpha}^2 + 3}{2};\ \frac{1}{2};\ \frac{1}{2}\gamma^2\right) \right\}$$

$$u(\gamma) = e^{-\frac{1}{2}\gamma^2} \left\{ A\gamma\ {}_1F_1\left(\frac{\bar{\alpha}^2 + 4}{2};\ \frac{3}{2};\ \frac{1}{2}\gamma^2\right) + B\ {}_1F_1\left(\frac{\bar{\alpha}^2 + 3}{2};\ \frac{1}{2};\ \frac{1}{2}\gamma^2\right) \right\} \qquad (5)$$

In order to satisfy the boundary condition $u(\infty) = 0$ we must know the asymptotic behavior of ${}_1F_1$.

According to reference 2, page 271, we have

$$
{}_1F_1(a;b;z) \approx \frac{(b-1)!}{(b-a-1)!}(-z)^{-a}\left\{1+\sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{a.\ .\ .(a+\nu-1)(a-b+1).\ .\ .(a-b+\nu)}{!z}\right\}+
\frac{(b-1)!}{(a-1)!}e^{z}z^{a-b}\left\{1+\sum_{\nu=1}^{\infty}\frac{(1-a).\ .\ .(\nu-a)(b-a).\ .\ .(b-a+\nu-1)}{\nu!\ z^{\nu}}\right\}+
$$

and, therefore,

$$
r\ {}_1F_1\left(\frac{\bar{a}^2+4}{2};\frac{3}{2};\frac{1}{2}r^2\right)\approx\frac{\frac{1}{2}!}{\left(-\frac{\bar{\alpha}^2+3}{2}\right)!}\left(-\frac{1}{2}\right)^{-\frac{\bar{\alpha}^2+4}{2}}r^{-(\bar{\alpha}^2+3)}
\left\{1+\sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{\frac{\bar{\alpha}^2+4}{2}.\ .\ .\left(\nu+\frac{\bar{\alpha}^2+2}{2}\right)\frac{\bar{\alpha}^2+3}{2}.\ .\ .\left(\nu+\frac{\bar{\alpha}^2+1}{2}\right)}{\nu!\left(\frac{1}{2}r^2\right)^{\nu}}\right\}
$$

$$
\frac{\frac{1}{2}\ !}{\frac{\bar{\alpha}^2+2}{2}\ !}e^{\frac{1}{2}r^2}\left(\frac{1}{2}\right)^{\frac{\bar{\alpha}^2+1}{2}}r^{\bar{\alpha}^2+2}
\left\{1+\sum_{\nu=1}^{\infty}\frac{\left(-\frac{\bar{\alpha}^2+2}{2}\right).\ .\ .\left(\nu-\frac{\bar{\alpha}^2+4}{2}\right)\left(-\frac{\bar{\alpha}^2+1}{2}\right).\ .\ .\left(\nu-\frac{\bar{\alpha}^2+3}{2}\right)}{\nu!\left(\frac{1}{2}r^2\right)^{\nu}}\right\}
$$

$$
{}_1F_1\left(\frac{\bar{\alpha}^2+3}{2};\frac{1}{2};\frac{1}{2}r^2\right)\approx\frac{\left(-\frac{1}{2}\right)!}{\left(-\frac{\bar{\alpha}^2+4}{2}\right)!}\left(-\frac{1}{2}\right)^{-\frac{\bar{\alpha}^2+3}{2}}r^{-(\bar{\alpha}^2+3)}
\left\{1+\sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{\frac{\bar{\alpha}^2+3}{2}.\ .\ .\left(\nu+\frac{\bar{\alpha}^2+1}{2}\right)\frac{\bar{\alpha}^2+4}{2}.\ .\ .\left(\nu+\frac{\bar{\alpha}^2+2}{2}\right)}{\nu!\left(\frac{1}{2}r^2\right)^{\nu}}\right\}+
\qquad (6)
$$

$$
\frac{\left(-\frac{1}{2}\right)!}{\frac{\bar{\alpha}^2+1}{2}\ !}e^{\frac{1}{2}r^2}\left(\frac{1}{2}\right)^{\frac{\bar{\alpha}^2+2}{2}}r^{\bar{\alpha}^2+2}
\left\{1+\sum_{\nu=1}^{\infty}\frac{\left(-\frac{\bar{\alpha}^2+1}{2}\right).\ .\ .\left(\nu-\frac{\bar{\alpha}^2+3}{2}\right)\left(-\frac{\bar{\alpha}^2+2}{2}\right).\ .\ .\left(\nu-\frac{\bar{\alpha}^2+4}{2}\right)}{\nu!\left(\frac{1}{2}r^2\right)^{\nu}}\right\}
$$

Since the corresponding series with the factor $e^{\gamma^2/2}$ of the two expansions (eq. (6)) are identical, we have

$$\lim_{\gamma\to\infty} u(\gamma) = \lim_{\gamma\to\infty} e^{-\frac{1}{2}\gamma^2} \left\{ A\gamma \; {}_1F_1\left(\frac{\bar{\alpha}^2+4}{2}; \frac{3}{2}; \frac{1}{2}\gamma^2\right) + B \; {}_1F_1\left(\frac{\bar{\alpha}^2+3}{2}; \frac{1}{2}; \frac{1}{2}\gamma^2\right) \right\}$$

$$= \left\{ A \; \frac{\frac{1}{2}\,!}{\frac{\bar{\alpha}^2+2}{2}\,!} \left(\frac{1}{2}\right)^{\frac{\bar{\alpha}^2+1}{2}} + B \; \frac{\left(-\frac{1}{2}\right)!}{\frac{\bar{\alpha}^2+1}{2}\,!} \left(\frac{1}{2}\right)^{\frac{\bar{\alpha}^2+2}{2}} \right\} \lim_{\gamma\to\infty} \gamma^{\bar{\alpha}^2+2} \left\{1 + o(1)\right\}$$

By setting the first brackets equal to zero, that is, by determining B as

$$B = -\frac{1}{\sqrt{2}} \frac{\frac{\bar{\alpha}^2+1}{2}\,!}{\frac{\bar{\alpha}^2+2}{2}\,!} A \tag{7}$$

we obtain

$$\lim_{\gamma\to\infty} u(\gamma) = 0$$

Thus, the solution of the first differential equation of equations (4) under the single boundary condition $u(\infty) = 0$ is

$$u(\gamma) = A\cdot e^{-\frac{1}{2}\gamma^2} \left\{ \gamma \; {}_1F_1\left(\frac{\bar{\alpha}^2+4}{2}; \frac{3}{2}; \frac{1}{2}\gamma^2\right) - \frac{1}{\sqrt{2}} \frac{\frac{\bar{\alpha}^2+1}{2}\,!}{\frac{\bar{\alpha}^2+2}{2}\,!} \; {}_1F_1\left(\frac{\bar{\alpha}^2+3}{2}; \frac{1}{2}; \frac{1}{2}\gamma^2\right) \right\} \tag{8}$$

Therefore,

$$u(\gamma) = A\cdot {}_0(\bar{\alpha}) e^{-\frac{1}{2}\gamma^2} \frac{1}{\gamma^{\bar{\alpha}^2+3}} \left\{1 + o(1)\right\}$$

III. THE GENERAL SOLUTION $v(\gamma)$

For the general solution of the second equation of equations (4) it is necessary to replace $(\bar{\alpha}^2 + 2)$ by $(\bar{\alpha}^2 - 1)$ in equation (5),

$$g(\gamma) = e^{-\frac{1}{2}\gamma^2}\left\{C\gamma\ {}_1F_1\left(\frac{\bar{\alpha}^2+1}{2};\ \frac{3}{2};\ \frac{1}{2}\gamma^2\right) + D\ {}_1F_1\left(\frac{\bar{\alpha}^2}{2};\ \frac{1}{2};\ \frac{1}{2}\gamma^2\right)\right\} \tag{9}$$

and with this $g(\gamma)$ the differential equation

$$v'' - \bar{\alpha}^2 = g + u' \tag{10}$$

is to be solved under the boundary conditions

$$v(\gamma) = o(\gamma)$$

$$v'(\infty) = 0$$

This will be done by the variation of constants method.

A fundamental system of the homogeneous part of equation (10) is $e^{\bar{\alpha}\gamma}$, $e^{-\bar{\alpha}\gamma}$. The general solution of differential equation (10) is, therefore,

$$v(\gamma) = K\cdot e^{\bar{\alpha}\gamma} + Le^{-\bar{\alpha}\gamma} + c_1(\gamma)e^{\bar{\alpha}\gamma} + c_2(\gamma)e^{-\bar{\alpha}\gamma}$$

where

$$c_1(\gamma) = -\int_{\gamma o}^{\gamma} \frac{g+u'}{W}\, e^{-\bar{\alpha}t}\, dt$$

$$c_2(\gamma) = \int_{\gamma o}^{\gamma} \frac{g+u'}{W}\, e^{\bar{\alpha}t}\, dt$$

and the Wronskian is $W = -2\alpha$; hence,

$$v(\gamma) = Ke^{\bar{\alpha}\gamma} + Le^{-\bar{\alpha}\gamma} + \frac{1}{2\bar{\alpha}}\, e^{\bar{\alpha}\gamma}\int_{\gamma o}^{\gamma} (g+u')e^{-\bar{\alpha}\gamma}\, dt\ -$$

$$\frac{1}{2\bar{\alpha}}\, e^{-\alpha\gamma}\int_{\gamma o}^{\gamma} (g+u')e^{\alpha\gamma}\, dt \tag{11}$$

We investigate the asymptotic behavior of solution (11). For this purpose we first require $u'(\gamma)$, which is computed by the formula (see sec. IX). If

$$f(\gamma) = e^{-\frac{1}{2}\gamma^2}\left\{\rho\gamma\ {}_1F_1\left(a;\ \frac{3}{2};\frac{1}{2}\gamma^2\right) - \sigma\ {}_1F_1\left(a-\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2}\gamma^2\right)\right\} \tag{12}$$

(ρ, σ constants), then

$$\frac{df}{d\gamma} = -\, e^{-\frac{1}{2}\gamma^2} \left\{ 2\sigma(a-1)\gamma \; {}_1F_1\left(a - \frac{1}{2};\, \frac{3}{2};\, \frac{1}{2}\gamma^2\right) - \rho \; {}_1F_1\left(a - 1;\, \frac{1}{2};\, \frac{1}{2}\gamma^2\right) \right\}$$

According to equation (8) we have

$$u'(\gamma) = -Ae^{-\frac{1}{2}\gamma^2} \left\{ \sqrt{2}\, \frac{\frac{\bar{\alpha}^2 + 1}{2}}{\frac{\bar{\alpha}^2}{2}} \gamma \; {}_1F_1\left(\frac{\bar{\alpha}^2 + 3}{2};\, \frac{3}{2};\, \frac{1}{2}\gamma^2\right) - {}_1F_1\left(\frac{\bar{\alpha}^2 + 2}{2};\, \frac{1}{2};\, \frac{1}{2}\gamma^2\right) \right\}$$

hence

$$u'(\gamma) = -\, Ae^{-\frac{1}{2}\gamma^2} \lambda_1(\bar{\alpha}) \frac{1}{\gamma\bar{\alpha}^2 + 2} \quad 1 + o(1) \qquad ^5$$

[5] Moreover, we have $\lambda_0(\bar{\alpha}) \equiv \lambda_1(\bar{\alpha})$.

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Further,

$$\gamma\ {}_1F_1\left(\frac{\bar{\alpha}^2+1}{2};\ \frac{3}{2};\ \frac{1}{2}\gamma^2\right)\approx\frac{\frac{1}{2}\,!}{\frac{\bar{\alpha}^2}{2}\,!}\left(-\frac{1}{2}\right)^{\frac{\bar{\alpha}^2+1}{2}}\gamma^{-\bar{\alpha}^2}$$

$$\left\{1+\sum_{\nu=1}^{\infty}(-1)\ \frac{\frac{\bar{\alpha}^2+1}{2}\ldots\left(\nu+\frac{\bar{\alpha}^2-1}{2}\right)\frac{\bar{\alpha}^2}{2}\ldots\left(\nu+\frac{\bar{\alpha}^2-2}{2}\right)}{\nu!\left(\frac{1}{2}\gamma^2\right)^{\nu}}\right\}+$$

$$\frac{\frac{1}{2}\,!}{\frac{\bar{\alpha}^2-1}{2}\,!}\ e^{\frac{1}{2}\gamma^2}\left(\frac{1}{2}\right)^{\frac{\bar{\alpha}^2-2}{2}}\gamma^{\bar{\alpha}^2-1}$$

$$\left\{1+\sum_{\nu=1}^{\infty}\frac{\left(-\frac{\bar{\alpha}^2-1}{2}\right)\ldots\left(\nu-\frac{\bar{\alpha}^2+1}{2}\right)\left(-\frac{\bar{\alpha}^2-2}{2}\right)\ldots\left(\nu-\frac{\bar{\alpha}^2}{2}\right)}{\nu!\left(\frac{1}{2}\gamma^2\right)^{\nu}}\right\}$$

$${}_1F_1\left(\frac{\bar{\alpha}^2}{2};\ \frac{1}{2};\ \frac{1}{2}\gamma^2\right)\approx\frac{\left(-\frac{1}{2}\right)!}{\left(-\frac{\bar{\alpha}^2+1}{2}\right)!}\left(-\frac{1}{2}\right)^{-\frac{\bar{\alpha}^2}{2}}\gamma^{-\bar{\alpha}^2}$$

$$\left\{1+\sum_{\nu=1}^{\infty}(-1)\ \frac{\frac{\bar{\alpha}^2}{2}\ldots\left(\nu+\frac{\bar{\alpha}^2-2}{2}\right)\frac{\bar{\alpha}^2+1}{2}\ldots\left(\nu+\frac{\bar{\alpha}^2-1}{2}\right)}{\nu!\left(\frac{1}{2}\gamma^2\right)^{\nu}}\right\}+$$

$$\frac{\left(-\frac{1}{2}\right)!}{\frac{\bar{\alpha}^2-2}{2}\,!}\ e^{\frac{1}{2}\gamma^2}\left(\frac{1}{2}\right)^{\frac{\bar{\alpha}^2-1}{2}}\gamma^{\bar{\alpha}^2-1}$$

$$\left\{1+\sum_{\nu=1}^{\infty}\frac{\left(-\frac{\bar{\alpha}^2-2}{2}\right)\ldots\left(\nu-\frac{\bar{\alpha}^2}{2}\right)\left(-\frac{\bar{\alpha}^2-1}{2}\right)\ldots\left(\nu-\frac{\bar{\alpha}^2+1}{2}\right)}{\nu!\left(\frac{1}{2}\gamma^2\right)^{\nu}}\right\}$$

Thus, the asymptotic integration of the sum $(g + u')$ occurring under the integrands in equation (11) is

$$g + u' = \lambda_2(\bar{\alpha},A,C,D)\,\frac{1}{\gamma^{1-\bar{\alpha}^2}}\left\{1 + o(1)\right\}$$

where the constants C and D are still undetermined.

Since the integral $\int_{\gamma o}^{\infty}(g + u')e^{-\bar{\alpha}t}\,dt$ exists for every value $\bar{\alpha} > 0$ we can replace expression (11) with

$$v(\gamma) = \left\{K + \frac{1}{2\bar{\alpha}}\int_{\gamma o}^{\infty}(g + u')e^{-\bar{\alpha}t}\,dt\right\}e^{\bar{\alpha}\gamma} + Le^{-\bar{\alpha}\gamma} -$$

$$\frac{1}{2\bar{\alpha}}\,e^{\bar{\alpha}\gamma}\int_{\gamma}^{\infty}(g + u')e^{-\bar{\alpha}t}\,dt - \frac{1}{2\bar{\alpha}}\,e^{-\bar{\alpha}\gamma}\int_{\gamma o}^{\gamma}(g + u')e^{\bar{\alpha}t}\,dt \quad (13)$$

IV. SATISFYING THE BOUNDARY CONDITIONS FOR $0 < \bar{\alpha}^2 < 1$

We assert the following conditions: If in equation (13) we choose

$$K = -\frac{1}{2\bar{\alpha}}\int_{\gamma o}^{\infty}(g + u')e^{-\bar{\alpha}t}\,dt$$

we cause the term with $e^{\bar{\alpha}\gamma}$ to drop out. The solution $v(\gamma)$ thereby determined satisfies the boundary conditions $v(\gamma) = o(\gamma)$, and $v'(\gamma) \to 0$ for $\gamma \to \infty$, if, in addition, $0 < \bar{\alpha}^2 < 1$.

Namely, we have

$$\left|e^{\bar{\alpha}\gamma}\int_{\gamma}^{\infty}(g + u')e^{-\bar{\alpha}t}\,dt\right| < \left|e^{\bar{\alpha}\gamma}(g + u')_{t=\gamma}\int_{\gamma}^{\infty}e^{-\bar{\alpha}\sigma}\,dt\right| =$$

$$\left|\frac{1}{\bar{\alpha}}(g + u')_{\gamma}\right| = o(\gamma),\ \text{if}\ \ 0 < \bar{\alpha}^2 < 2$$

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Further, we have

$$e^{-\bar{\alpha}\gamma}\int_{\gamma_0}^{\gamma}(g + u')e^{\bar{\alpha}t}\,dt = e^{-\bar{\alpha}\gamma}\int_{\gamma_0}^{\gamma}\left\{\lambda_2\frac{1}{t^{1-\bar{\alpha}^2}} + o\left(\frac{1}{t^{1-\bar{\alpha}^2}}\right)\right\}e^{\bar{\alpha}t}\,dt =$$

$$e^{-\bar{\alpha}\gamma}\int_{\gamma_0}^{\gamma}\lambda_2\frac{1}{t^{1-\bar{\alpha}^2}}e^{\bar{\alpha}t}\,dt\,\{1 + o(1)\} =$$

$$\lambda_2 e^{-\bar{\alpha}\gamma}\left\{\frac{1}{\bar{\alpha}}e^{\bar{\alpha}\gamma}\frac{1}{\gamma^{1-\bar{\alpha}^2}} - \frac{1}{\bar{\alpha}}e^{\bar{\alpha}\gamma_0}\frac{1}{\gamma_0^{1-\bar{\alpha}^2}} - \frac{\bar{\alpha}^2 - 1}{\bar{\alpha}}\int_{\gamma_0}^{\gamma}\frac{1}{t^{2-\bar{\alpha}^2}}e^{\bar{\alpha}t}\right\}dt\,\{1 + o(1)\} =$$

$$\lambda_2(\bar{\alpha},A,C,D)\frac{1}{\bar{\alpha}}\frac{1}{\gamma^{1-\bar{\alpha}^2}}\{1 + o(1)\} = o(\gamma) \quad \text{for} \quad 0 < \bar{\alpha}^2 < 2$$

As we have just shown, the solution

$$v(\gamma) = Le^{-\bar{\alpha}\gamma} - \frac{1}{2\bar{\alpha}}e^{\bar{\alpha}\gamma}\int_{\gamma}^{\infty}(g + u')e^{-\bar{\alpha}t}\,dt - \frac{1}{2\bar{\alpha}}e^{-\bar{\alpha}\gamma}\int_{\gamma_0}^{\gamma}(g + u')e^{\bar{\alpha}t}\,dt \tag{14}$$

satisfied the first boundary condition. We have

$$v'(\gamma) = -\bar{\alpha}Le^{-\bar{\alpha}\gamma} - \frac{1}{2}e^{\bar{\alpha}\gamma}\int_{\gamma}^{\infty}(g + u')e^{-\bar{\alpha}t}\,dt + \frac{1}{2}e^{-\bar{\alpha}\gamma}\int_{\gamma_0}^{\gamma}(g + u')e^{\bar{\alpha}t}\,dt$$

With the same considerations as given previously, the second boundary condition $v'(\infty) = 0$ is satisfied for $0 < \bar{\alpha}^2 < 1$.

V. SATISFYING THE BOUNDARY CONDITIONS FOR $\bar{\alpha}^2 \geq 1$

If $\bar{\alpha}^2 \geq 1$, the two boundary conditions for $v(\gamma)$ cannot be satisfied without determining another of the free constants C, D, and L. For this purpose in equation (9) we must drop the part of the asymptotic development of g not decreasing as $e^{-\gamma^2/2}$ by determining C as a function of D analogously to what was done in deriving equation (7). This is attained through

$$C = -D\sqrt{2}\,\frac{\left(\frac{\bar{\alpha}^2 - 1}{2}\right)!}{\left(\frac{\bar{\alpha}^2 - 2}{2}\right)!}$$

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$$\varphi_1' = \varphi_2$$

$$\varphi_2' = -F\varphi_2 + (2F' + \overline{\alpha}^2)\varphi_1 - F''\varphi_3$$

$$\varphi_3' = \varphi_4$$

$$\varphi_4' = \varphi_5$$

$$\varphi_5' = \varphi_6$$

$$\varphi_6' = -F\varphi_6 - (F' - 2\overline{\alpha}^2)\varphi_5 + (\overline{\alpha}^2 F - F'')\varphi_4\ \overline{\alpha}^2(\overline{\alpha}^2 - F')\varphi + F'''\ \varphi_3 + \qquad (15)$$
$$2F'\varphi_2 + 2F''\ {}_1\varphi_1$$

$$\varphi_1 = u \qquad \varphi_3 = v \qquad \varphi_1(0) = 0$$

$$\varphi_2 = u' \qquad \varphi_4 = v' \qquad \varphi_3(0) = 0$$

$$\varphi_5 = v'' \qquad \varphi_4(0) = 0$$

$$\varphi_6 = v'''$$

Owing to the continuity of the coefficients of this system in $0 \leq \eta \leq \eta_o$, the Lipschitz condition is entirely satisfied in this interval; therefore, six linearly independent fundamental solutions of the system (15) exist in this interior region.

In order to connect the outer and inner solutions at $\eta = \eta_o$, it is required that $(\eta_o = \gamma_o + c)$

$$u_i(\eta_o) = u_a(\eta_o) \qquad v_i(\eta_o) = v_a(\eta_o)$$

$$u_i'(\eta_o) = u_a'(\eta_o) \qquad v_i'(\eta_o) = v_a'(\eta_o)$$

$$v_i''(\eta_o) = v_a''(\eta_o)$$

$$v_i'''(\eta_o) = v'''_a(\eta_o)$$

The requirements of the connecting outer and inner solutions and the as yet unsatisfied boundary conditions for the inner solutions at $\eta = 0$ represent nine conditions. These are nine homogeneous equations for the six constants of the inner solutions and the still free constants of the outer solutions.

Let us again consider the interval $0 < \overline{\alpha}^2 < 1$. For each choice of $\overline{\alpha}^2$ in this interval four free constants (A, C, D, and L) are available from the external solutions. Thus, for each $\overline{\alpha}^2$ in $0 < \overline{\alpha}^2 < 1$ we have to satisfy nine homogeneous equations for a total of ten unknowns; this is always possible. In this way values $0 < \overline{\alpha}^2 < 1$ form a continuum of eigenvalues.[6]

The question now arises whether there are also eigenvalues among the values $\overline{\alpha}^2 \geq 1$. We must modify our considerations adduced in the preceding paragraphs because we only have three further constants of the external solutions at our disposal. This leads to nine homogeneous equations for nine unknowns. The question as to the solubility of this system, without the aid of numerical methods, can only be answered if we know a fundamental system of equation (15). For this reason, our eigenvalue problem was attacked numerically.

VII. NUMERICAL INVESTIGATIONS WITH THE Z 4 COMPUTER

The Institute for Applied Mathematics of the E. T. H. Zürich (directed by Dr. E. Stiefel) has kindly taken over the problem. Under the guidance of Dr. Mähly, the interval $0 < \overline{\alpha}^2 \leq 5$ was investigated with the aid of the program-controlled Z 4 computer. The computation on the Z 4 was preceded by further extensive theoretical investigations under Dr. Mähly, whom I also thank for his suggestions in regard to the preceding considerations.

Since a more accurate presentation of the work done in Zürich would extend far beyond the limitations of the present report, we restrict ourselves here to a brief summary and mentioning of the results achieved.

Starting with $\eta = 5$, the integration of differential equations (1) toward zero was begun with steps of 0.2. The value $\eta = 5$ was found favorable for the starting point of the integration because a value of η was to be chosen at which the asymptotic properties and the solutions of the differential equations are satisfied with sufficient accuracy, while the integration toward zero involves small errors.

As previously shown, three of the exponentially rising fundamental solutions are available. These solutions are combined linearly in such a manner that $u(0) = v(0) = 0$. For $u \equiv 0$, $v \equiv 0$, and $g \equiv 0$ not to be obtained from the differential equations (1), at least one of the

[6]On the basis of a heuristic consideration Dr. Mähly first expresed the theory that all $0 < \overline{\alpha}^2 < 1$ values are eigenvalues.

three values $g(0)$, $g'(0)$, and $u'(0)$ must be different from zero. Accordingly, in order to satisfy the third boundary condition $v'(0)$, at least one of the three magnitudes

$$\frac{v'(0)}{g(0)}, \frac{v'(0)}{g'(0)}, \frac{v'(0)}{u'(0)}$$ [7]

must vanish. These tests were conducted by repeatedly carrying out the computation for variously chosen values of $\bar{\alpha}^2$.

The evaluation showed that no eigenvalues exist in the interval $0 < \bar{\alpha}^2 < 5$ if only fundamental solutions, which drop exponentially toward infinity, are admitted. The region $\bar{\alpha}^2 > 5$ has not yet been investigated.[8] As shown previously, the continuously distributed eigenvalues in $0 < \bar{\alpha}^2 < 1$ give a weakly decreasing power solution in addition to these solutions falling exponentially toward infinity.

VIII. UNSTABLE DISTURBANCES

The question arises whether in addition to the neutral disturbances $(\bar{\beta} = 0)$ thus far considered, unstable disturbances $(\bar{\beta} > 0)$ also exist. The differential equations of such unstable disturbances are obtained from equation (1) if the operator

$$L = \frac{d}{d\eta}\left\{\frac{d}{d\eta} + F\right\} - (\bar{\beta} + \bar{\alpha}^2)$$

is applied. With the same considerations that we have adduced for the case $\bar{\beta} = 0$, it can be shown that solutions exist for all values $\bar{\alpha} > 0$ and $\bar{\beta} > 0$ for which $0 < \bar{\beta} + \bar{\alpha}^2 < 1$, so that all these values form eigenvalue pairs. However, nothing can be said as to the existence of such eigenvalue pairs in the interval $\bar{\beta} + \bar{\alpha}^2 \geq 1$. For this reason the numerical computations must be carried out anew in a suitable manner.[9]

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[7]The introduction of these fractions was found very useful in setting up the computer program.

[8]From physical considerations, additional eigenvalues for $\bar{\alpha}^2 > 5$ are hardly to be expected. (This means very small vortex dimensions.)

[9]Similarly, for damped disturbances $(\bar{\beta} < 0)$ all values $0 < \bar{\beta} + \bar{\alpha}^2 < 1$ form eigenvalue pairs. However, further results that are only of small interest can here be stated.

IX. PROOF OF FORMULA (12)

If

$$f(\gamma) = e^{-\frac{1}{2}\gamma^2}\left\{\rho\gamma \; {}_1F_1\left(a; \tfrac{3}{2}; \tfrac{1}{2}\gamma^2\right) - \sigma \; {}_1F_1\left(a - \tfrac{1}{2}; \tfrac{1}{2}; \tfrac{1}{2}\gamma^2\right)\right\}$$

then

$$\frac{df}{d\gamma} = -e^{-\frac{1}{2}\gamma^2}\left\{2\sigma(\alpha - 1)\gamma \; {}_1F_1\left(a - \tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{2}\gamma^2\right) - \rho \; {}_1F_1\left(a - 1; \tfrac{1}{2}; \tfrac{1}{2}\gamma^2\right)\right\}$$

From the series

$${}_1F_1(a;b;z) = 1 + \frac{a}{b} \cdot z + \frac{a(a + 1)}{b(b + 1)} \frac{z^2}{2!} + \cdot \cdot \cdot$$

it is seen that

$$\frac{d}{dz} \; {}_1F_1(a;b;z) = \frac{a}{b} \; {}_1F_1(a + 1;\; b + 1;\; z)$$

so that

$$\frac{d}{d\gamma} \; {}_1F_1\left(a;\; b;\; \tfrac{1}{2}\gamma^2\right) = \gamma \frac{a}{b} \; {}_1F_1\left(a + 1;\; b + 1;\; \tfrac{1}{2}\gamma^2\right)$$

We then have

$$\frac{df}{d\gamma} = e^{-\frac{1}{2}\gamma^2}\left\{-\rho\gamma^2 \; {}_1F_1\left(a; \tfrac{3}{2}; \tfrac{1}{2}\gamma^2\right) + \sigma\gamma \; {}_1F_1\left(a - \tfrac{1}{2}; \tfrac{1}{2}; \tfrac{1}{2}\gamma^2\right) + \right.$$

$$\rho \; {}_1F_1\left(a; \tfrac{3}{2}; \tfrac{1}{2}\gamma^2\right) + \rho\gamma^2 \frac{2a}{3} \; {}_1F_1\left(a + 1; \tfrac{5}{2}; \tfrac{1}{2}\gamma^2\right) -$$

$$\left. \sigma\gamma(2a - 1) \; {}_1F_1\left(a + \tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{2}\gamma^2\right)\right\}$$

Using the recursion formula (ref. 2, p. 271, 3)

$$a \; {}_1F_1(a + 1;\; b + 1;\; z) = (a - b) \; {}_1F_1(a;\; b + 1;\; z) + b \; {}_1F_1(a;\; b;\; z)$$

$b = \frac{3}{2}$:

$$\frac{2a}{3} \; {}_1F_1\left(a + 1; \tfrac{5}{2}; \tfrac{1}{2}\gamma^2\right) - {}_1F_1\left(a; \tfrac{3}{2}; \tfrac{1}{2}\gamma^2\right) = \left(\frac{2a}{3} - 1\right) {}_1F_1\left(a; \tfrac{5}{2}; \tfrac{1}{2}\gamma^2\right)$$

replace a by (a - 1/2), so that

$b = \frac{1}{2}$:

$$
{}_1F_1\left(a - \frac{1}{2}; \frac{1}{2}; \frac{1}{2}\gamma^2\right) - (2a - 1)\,{}_1F_1\left(a + \frac{1}{2}; \frac{3}{2}; \frac{1}{2}\gamma^2\right) = 2(1 - a)\,{}_1F_1\left(a - \frac{1}{2}; \frac{3}{2}; \frac{1}{2}\gamma^2\right)
$$

Thus

$$
\frac{df}{d\gamma} = e^{-\frac{1}{2}\gamma^2}\left\{\rho\gamma^2\left(\frac{2a}{3} - 1\right){}_1F_1\left(a; \frac{5}{2}; \frac{1}{2}\gamma^2\right) + \sigma\gamma 2(1 - a)\,{}_1F_1\left(a - \frac{1}{2}; \frac{3}{2}; \frac{1}{2}\gamma^2\right) + \rho\ {}_1F_1\left(a; \frac{3}{2}; \frac{1}{2}\gamma^2\right)\right\}
$$

In the recursion formula (ref. 2, p. 271)

$$
z\ {}_1F_1(a + 1; b + 1; z) = b\ {}_1F_1(a + 1; b; z) - b\ {}_1F_1(a; b; z)
$$

replace a by (a - 1), so that

$b = \frac{3}{2}$:

$$
\frac{1}{2}\gamma^2\ {}_1F_1\left(a; \frac{5}{2}; \frac{1}{2}\gamma^2\right) - \frac{3}{2}\ {}_1F_1\left(a; \frac{3}{2}; \frac{1}{2}\gamma^2\right) = -\frac{3}{2}\ {}_1F_1\left(a - 1; \frac{3}{2}; \frac{1}{2}\gamma^2\right)
$$

$$
\gamma^2\left(\frac{2a}{3} - 1\right){}_1F_1\left(a; \frac{5}{2}; \frac{1}{2}\gamma^2\right) + {}_1F_1\left(a; \frac{3}{2}; \frac{1}{2}\gamma^2\right) = (2a - 2)\,{}_1F_1\left(a; \frac{3}{2}; \frac{1}{2}\gamma^2\right) - (2a - 3)\,{}_1F_1\left(a - 1; \frac{3}{2}; \frac{1}{2}\gamma^2\right)
$$

In the recursion formula (ref. 2, p. 271, 3)

$$
a\ {}_1F_1(a + 1; b + 1; z) = (a - b)\,{}_1F_1(a; b + 1; z) + b\ {}_1F_1(a; b; z)
$$

replace a by (a - 1), so that

$b + 1 = \frac{3}{2}$:

$$
(2a - 2)\,{}_1F_1\left(a; \frac{3}{2}; \frac{1}{2}\gamma^2\right) - (2a - 3)\,{}_1F_1\left(a - 1; \frac{3}{2}; \frac{1}{2}\gamma^2\right) = {}_1F_1\left(a - 1; \frac{1}{2}; \frac{1}{2}\gamma^2\right)
$$

Thus

$$\frac{df}{d\gamma} = e^{-\frac{1}{2}\gamma^2}\left\{\rho\ {}_1F_1\left(a - 1; \frac{1}{2}; \frac{1}{2}\gamma^2\right) - 2\sigma(a - 1)\gamma\ {}_1F_1\left(a - \frac{1}{2}; \frac{3}{2}; \frac{1}{2}\gamma^2\right)\right\} =$$

$$-e^{-\frac{1}{2}\gamma^2}\left\{2\sigma(a - 1)\gamma\ {}_1F_1\left(a - \frac{1}{2}; \frac{3}{2}; \frac{1}{2}\gamma^2\right) - \rho\ {}_1F_1\left(a - 1; \frac{1}{2}; \frac{1}{2}\gamma^2\right)\right\}$$

The correctness of formula (12) has thus been demonstrated.

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